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D-branes on vanishing del Pezzo surfaces

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Abstract: We analyze in detail the case of a marginally stable D-Brane on a collapsed del Pezzo surface in a Calabi-Yau threefold using the derived category of quiver representations and the idea of aligned gradings. We show how the derived category approach to D-branes provides a straight-forward and rigorous construction of quiver gauge theories associated to such singularities. Our method shows that a procedure involving exceptional collections used elsewhere in the literature is only valid if some tachyon-inducing Ext^3 groups are zero. We then analyze in generality a large class of Seiberg dualities which arise from tilting equivalences. It follows that some (but not all) mutations of exceptional collections induce Seiberg duality in this context. The same tilting equivalence can also be used to remove unwanted Ext^3 groups and convert an unphysical quiver into a physical one.

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1. Introduction

Suppose a D-brane is marginally stable against decay into a collection of stable "constituent" or "fractional" D-branes. Each constituent D-brane may appear with multiplicity N_i and so is associated to a factor of $\mathrm{U}(N_i)$ in the world-volume gauge theory. The fact that these D-branes are marginally bound implies that there are massless open strings between them which correspond to chiral fields in $(\overline{\mathbf{N}}_i, \mathbf{N}_j)$ representations in the above gauge theory. In this way we associate a "quiver" gauge theory to any D-brane decay.

For D-branes on a Calabi-Yau threefold, we expect an enormous number (probably dense) of walls of marginal stability in the moduli space and so we should have a correspondingly huge number of possibilities for quiver gauge theories.

The case most frequently studied concerns a BPS B-type D-brane corresponding to a point on a Calabi-Yau space X. If we place such a D-brane at a "singular" point we expect to possibly find a marginal decay. The best-understood case probably concerns orbifold singularities locally of the form $\mathbb{C}^3/G$ [1]–[5]. The D-brane decays into a set of "fractional branes" associated to irreducible representations of G and the associated quiver is given by the McKay quiver.

We would like to consider the case of a complex surface S shrinking down to a point inside X to produce a singularity. Again, we would expect the D-brane associated to this singular point to decay into fractional branes. If S is smooth and irreducible it must be a del Pezzo surface. These cases overlap with the orbifolds only in the single case that $S = \mathbb{P}^2$ corresponding to $\mathbb{C}^3/\mathbb{Z}_3$. Other del Pezzo's do not produce orbifold singularities, and other orbifolds have exceptional divisors with more than one component.

This case has been discussed many times in the literature [6]–[14]. The essentially new thing we are going to do in this paper is to bring the full weight of the machinery of the derived category of coherent sheaves [15, 16, 5] to bear on the problem so that we can make a clear statement (see theorem 3). We believe this clarifies many aspects of this subject.

An important ingredient in the derived category approach is an integer grading. Given a D-brane A one may produce another D-brane $A[n]$ by "shifting A n-places to the left." If n is odd and A is the only D-brane under consideration then $A[n]$ is an "anti" D-brane. In other words we only care about n mod 2. When we have more than one D-brane the relative integer grading between branes becomes important and one cannot simply reduce mod 2. That is, it is too simplistic to talk in terms of branes and anti-branes. In this paper we will see exactly how the latter picture can go wrong.

One can attack the problem we are interested in by using the A-model description of the mirror. This may be done following the ideas of [17, 18, 11]. The mirror to the grading in the derived category concerns the degree of Floer cohomology as given by Maslov indices. Thus, reducing the grading mod 2 amounts to merely counting points of intersection between 3-cycles, rather than the more intricate procedure of computing the Floer cohomology groups. The derived category picture of B-branes is generally much easier to handle than Floer cohomology and the Fukaya category [19] of A-branes. It follows that mirror symmetry is not really a useful tool given the degree of precision we desire this paper.

One of the most interesting aspects of quiver gauge theories concerns "Seiberg dualities" [20]. Although originally considered as a thoroughly quantum effect in $N = 1$ field theories, it appears that these dualities correspond to equivalences between D-branes even in the limit of zero string coupling. The derived category picture of this story corresponds to "tilting equivalences," as proposed by Berenstein and Douglas [21]. The main obstacle to applying the full power of the mathematics of quiver representations to Seiberg duality is the occurrence of oriented loops in the quiver. If a quiver has an oriented loop, then the quiver representations become infinite-dimensional, and it becomes much harder to make specific statements (although an example was studied in [22]). Sadly, any physical quiver for the problems we are analyzing in this paper has these unwanted oriented loops.

In the case of del Pezzo surfaces, we may associate some of the arrows in the gauge quiver to the intrinsic properties of the surface itself, while the remaining arrows are associated to the embedding of this surface into the Calabi-Yau manifold. By deleting these latter arrows we remove all oriented loops. Thus we are able to analyze the problem using finite-dimensional representation theory. This is how we proceed in this paper. Fortunately, it is a simple matter to add in the arrows associated to the embedding after most of the analysis has been done.

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This means we are able to provide a fairly general analysis of Seiberg dualities for del Pezzo surfaces. An interesting fact that we will observe is that there are two different tilting equivalences, each the inverse of the other, associated to Seiberg duality. These tilts differ by whether the brane to anti-brane transformation is given by a shift [1] or a shift [−1]. For many nodes in the quiver only one of these two tilts produces a valid duality.

In section 2 we review the general picture of how a marginal decay yields a quiver gauge theory. Of particular interest is the way in which this occurs because of an alignment in the gradings of a large class of D-branes in the problem.

In section 3 we will review the mathematics of the derived category of coherent sheaves on a del Pezzo surface and its relation to quiver categories and exceptional collections of sheaves. This involves the relationship between tilting complexes and projective objects. We then show that the quiver associated to the derived category of the del Pezzo surface is indeed the gauge quiver for an object such as a marginally decaying 0-brane, so long as some Ext^3 groups vanish.

In section 4 we study how Seiberg dualities on the quiver gauge theory are given by tilting equivalences. We then show how the tilting picture can be tied in with the mutation picture for Seiberg duality which is used elsewhere in the literature. We also show how tilts may also remove the troublesome Ext^3's induced by some exceptional collections.

When this paper was completed, a paper appeared with a large overlap with the work presented here [23].

2. Marginal decays and quivers

Suppose we have a B-type BPS D-brane, A, on a Calabi-Yau manifold X. Let us assume it fills the spatial directions of uncompactified spacetime. Our convention will be to use the notation "n-brane" where n refers to the dimensionality of the brane *within the Calabi-Yau*. Thus a 0-brane, as we denote it, would correspond to a point on X and would be considered a 3-brane in the full ten-dimensional spacetime.

It is well-established [15, 16, 4, 5, 24], that the B-type topological D-branes on X are objects in $\mathbf{D}(X)$, the derived category of coherent sheaves on X. In order to correspond to a physical D-brane, such an object must be Π-stable [25, 15, 26]. This Π-stability condition depends both on the complexified Kähler form, $B + iJ$, of X and upon the position (i.e., moduli) of the D-brane in X.

Of particular interest will be the case where the D-brane is a 0-brane corresponding to a point $p \in X$. Generally speaking one expects such a D-brane to be stable if p corresponds to a smooth point. If p is at a singularity one might expect the 0-brane to decay.

The general picture for Π-stability proceeds as follows. Each stable B-brane has a "grade" $\xi \in \mathbb{R}$ which varies with $B + iJ$. The grading is defined mod 2 by the argument of the central charge:

$$\xi(A) = \frac{1}{\pi} \arg Z(A) \pmod 2 . \tag{2.1}$$

The mod 2 ambiguity can be fixed from the large radius limit [5]. ξ is not a single-valued function of the Kähler moduli but should be thought of as a function on the Teichmüller

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space of $B + iJ$. The Hilbert space of scalar objects in the D-brane world-volume corresponding to open strings from a B-brane B to another B-Brane A in the sector seen by the topological field theory is given by $\oplus_p \mathrm{Ext}^p(A, B)$. The mass of such an open string is given by [15]

$$m^2 = \frac{1}{2}(\xi(A) - \xi(B) + p - 1)\,. \tag{2.2}$$

If this string is tachyonic, then it will bind A to B. If it is massive, then, in the absence of any other bindings, A and B will not be bound. Thus if we move in the moduli space by varying $B + iJ$, the gradings ξ of each B-brane will vary and the spectrum of stable B-branes will jump as the masses of the above open strings pass through zero.

Suppose we are in a situation where the grades, ξ, of a set of B-branes $L_0, L_1, \ldots, L_{n-1}$ coincide. We will discuss examples of this shortly. Let us also assume the following relation is satisfied:

$$\mathrm{Hom}(L_i, L_j) = \mathbb{C}\delta_{ij}\,. \tag{2.3}$$

Since $\mathrm{Hom} = \mathrm{Ext}^0$, according to (2.2), there are no tachyons betweens these D-branes.[1] The map from L_i to itself gives rise to a vector particle and thus (classically) a U(1) gauge theory in the D-brane world-volume. As usual, if we have N_i B-branes of the type L_i, we would obtain a $\mathrm{U}(N_i)$ gauge theory.

According to (2.2) the massless scalars between L_i and L_j would be counted by $\mathrm{Ext}^1(L_i, L_j)$ and $\mathrm{Ext}^1(L_j, L_i)$. If we have N_i copies of each L_i, we would have $\mathrm{Ext}^1(L_i, L_j)$ scalar fields transforming in a $(\overline{\mathbf{N}}_i, \mathbf{N}_j)$ representation of the $\mathrm{U}(N_i) \times \mathrm{U}(N_j)$ part of the gauge group.

Therefore, when the grades of a set of B-branes coincide and satisfy (2.3), we automatically have a "quiver gauge theory." Each node in the quiver is labeled by i and corresponds to a $\mathrm{U}(N_i)$ factor of the gauge group. Each arrow in the diagram corresponds to an Ext^1 group and is interpreted as a bifundamental chiral field in the four-dimensional D-brane world volume theory.

This quiver gauge theory can be thought of as describing a marginal binding of the associated D-branes. The bifundamental chiral fields are exactly massless. A perturbation of $B + iJ$ may make these open strings tachyonic or massive, making the bound state stable or unstable, respectively.

The open strings $\mathrm{Ext}^p(L_i, L_j)$ for $p > 1$ correspond to very massive strings and will be ignored in the quiver gauge theory.

We would like to study marginally-stable B-branes corresponding to isolated points in a Calabi-Yau threefold. Presumably such an instability requires the point be a singularity. Since the stability is governed by $B + iJ$, the singularity must be obtained by a deformation of $B + iJ$, i.e., a blow-down of something inside the Calabi-Yau threefold. If the subspace that is blown-down to a point is a smooth irreducible surface S, then this surface must be a del Pezzo surface [27]. We will restrict attention to this case in this paper.

The case of $S = \mathbb{P}^2$ collapsing to a point corresponds to the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold. The relevant analysis for this problem has been studied in [28, 3, 2, 29, 5]. There it is established

[1] The tachyon from L_i to itself is removed by the GSO projection.

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that the three fractional branes into which the 0-brane decays indeed have exactly the same value for their grading when precisely at the orbifold point in the moduli space. Thus our picture applies. Any perturbation away from this point in the moduli space would destroy the marginal decay of the 0-brane into the 3 fractional branes. At least one bound state of two of the three fractional branes would be definitely stable or unstable.

The del Pezzo surfaces are $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ blown-up at m points, which we denote by dP_m. There are thus $\dim H^{\mathrm{even}} = m+3$ periods characterizing the central charges for branes on dP_m. There is an irrelevant overall scaling of the periods, so we have $m+2$ independent periods. The moduli space of $B+iJ$ has complex dimension $\dim H^2 = m+1$. Thus, in order to align the gradings of all the B-branes, i.e., align all the arg's of the periods, we would need to impose $m+2$ real constraints on $2m+2$ real moduli. Thus we expect a subspace of the moduli space of real dimension m for which the gradings are suitably aligned. This ties in with the case $m=0$ (the orbifold above), where there was only one point in the moduli space.

Our expectation is therefore that the alignment occurs somewhere in moduli space where the del Pezzo surface has shrunk to zero size and there may be remaining degrees of freedom in the moduli (given by B-fields) which do not effect this alignment. We show this explicitly for $\mathbb{P}^1 \times \mathbb{P}^1$ in the next section.

In a way, this alignment has already been demonstrated for any singularity that can be obtained by the partial resolution of an orbifold singularity. The McKay correspondence proves that gradings align at the orbifold point (theorem 4 of [5]). A partial resolution of this orbifold should preserve the alignment of the fractional branes associated with the remaining singularity. This covers a wide class of del Pezzo surfaces. It would be nice to make this argument rigorous.

2.1 Gradings and periods for $S = \mathbb{P}^1 \times \mathbb{P}^1$

In this section we find the subspace in the moduli space where the gradings are aligned for the example of $S = \mathbb{P}^1 \times \mathbb{P}^1$. As discussed in [5], the central charges of the B-branes on X are given by a period computation on the mirror of X. Which periods appear is determined by matching the form of the D-brane charge in the large radius limit:

$$Z(\mathscr{E}) = \int_X e^{-(B+iJ)}\, ch(\mathscr{E})\sqrt{td(X)}\,. \tag{2.4}$$

We will be concerned with branes on $S \subset X$. Let η_1, η_2 be elements in $H^2(S,\mathbb{Z})$ dual to the hyperplane classes of the $\mathbb{P}^1$'s. Expressing $B+iJ$ in this dual basis as $B+iJ = t_1\eta_1 + t_2\eta_2$, $Z(\mathscr{E})$ reduces to

$$Z(\mathscr{E}) = \int_S e^{-(t_1\eta_1+t_2\eta_2)}\, ch(\mathscr{E})\sqrt{\frac{td(S)}{td(N_S)}}, \tag{2.5}$$

where N_S denotes the normal bundle to S. In the large radius limit ($\mathrm{Im}(t_1), \mathrm{Im}(t_2) \to \infty$) where the above expression is valid, we find that $Z(\mathscr{E})$ has terms that scale as 1, t_1, t_2, and $t_1 t_2$. The relevant subspace of the moduli space of the non-linear σ-model on X is most conveniently parametrized by the B-model coordinates z_1, z_2, which are related to t_1, t_2 by

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the mirror map. Asymptotically, this is given by

$$t_1 \sim \frac{1}{2\pi i}\log(z_1)\,,$$
$$t_2 \sim \frac{1}{2\pi i}\log(z_2)\,. \tag{2.6}$$

More generally, t_1, t_2 are given by ratios of periods on the mirror. It is clear that $Z(\mathscr{E})$ will be expressible as a linear combination of four periods that in the large radius limit ($|z_1|, |z_2| \to 0$) scale as

$$\begin{aligned}\Phi_0 &\sim 1\,,\\ \Phi_1 &\sim \frac{1}{2\pi i}\log(z_1)\,,\\ \Phi_2 &\sim \frac{1}{2\pi i}\log(z_2)\,,\\ \Phi_3 &\sim -\frac{1}{4\pi^2}\log(z_1)\log(z_2)\,.\end{aligned} \tag{2.7}$$

The gradings will align for those (z_1, z_2) where these periods are simultaneously real. To find these points in the moduli space, we must first calculate these periods. We do this by finding solutions to the Picard-Fuchs equations with appropriate asymptotics. In the case of toric S, it is known that the Picard-Fuchs system is a special case of the GKZ system, and for $S = \mathbb{P}^1 \times \mathbb{P}^1$, the Picard-Fuchs system is given by

$$\begin{aligned}\left(\theta_1^2 - 4z_1\left(\theta_1+\theta_2+1\right)\left(\theta_1+\theta_2\right)\right)\Phi(z_1,z_2) &= 0\,,\\ \left(\theta_2^2 - 4z_2\left(\theta_1+\theta_2+1\right)\left(\theta_1+\theta_2\right)\right)\Phi(z_1,z_2) &= 0\,,\end{aligned} \tag{2.8}$$

where $\theta_a = z_a\frac{\partial}{\partial z_a}$.

By using the techniques of [28, 30] we find the following solutions:

$$\begin{aligned}\Phi_0(z_1,z_2) &= 1\,,\\ \Phi_1(z_1,z_2) &= \frac{1}{2\pi i}\log(e^{i\pi}z_1) + \frac{1}{i\pi}\sum_{(m,n)\neq(0,0)} A_{mn}z_1^m z_2^n\,,\\ \Phi_2(z_1,z_2) &= \frac{1}{2\pi i}\log(e^{i\pi}z_2) + \frac{1}{i\pi}\sum_{(m,n)\neq(0,0)} A_{mn}z_1^m z_2^n\,,\\ \Phi_3(z_1,z_2) &= -\frac{1}{4\pi^2}\Bigg(\log(e^{i\pi}z_1)\log(e^{i\pi}z_2) + \\ &\qquad + \sum_{(m,n)\neq(0,0)} A_{mn}z_1^m z_2^n\left(\log(e^{i\pi}z_1) + \log(e^{i\pi}z_2) + \chi_{mn}\right)\Bigg)\,,\end{aligned} \tag{2.9}$$

where

$$\begin{aligned}A_{mn} &= \frac{\Gamma(2m+2n)}{\Gamma(m+1)^2\Gamma(n+1)^2}\,,\\ \chi_{mn} &= 4\Psi(2m+2n) - 2\Psi(m+1) - 2\Psi(n+1)\,,\end{aligned} \tag{2.10}$$

$\Psi(z)$ is the usual di-Gamma function, and the sum $\sum_{(m,n)\neq(0,0)}$ runs over all non-negative

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m, n with the exception of $m = n = 0$. The power series converge for $|z_1| < \frac{1}{4}$ and $|z_2| < \frac{1}{4}$. These radii of convergence are determined by the distance of the large radius limit point to the discriminant locus.

Having found the appropriate solutions in the large radius limit, we can analytically continue these periods to a phase where S shrinks by using Mellin-Barnes representations of these solutions. We continue to a phase where $|z_2| \to \infty$, and the appropriate coordinates are given by $y_1 = z_1/z_2$, and $y_2 = 1/z_2$. Denoting by $\widetilde{\Phi}_i$ the analytic continuation of Φ_i, we find

$$\begin{aligned}
\widetilde{\Phi}_0(y_1, y_2) &= 1\,, \\
\widetilde{\Phi}_1(y_1, y_2) &= \frac{1}{2\pi i}\log(y_1) + \widetilde{\Phi}_2(y_1, y_2)\,, \\
\widetilde{\Phi}_2(y_1, y_2) &= -\frac{1}{2\pi i}\left(e^{-i\pi}y_2\right)^{\frac{1}{2}} \sum_{m,n\geq 0} B_{mn} y_1^m y_2^n\,, \\
\widetilde{\Phi}_3(y_1, y_2) &= \frac{1}{12} - \frac{1}{4\pi^2}\left(e^{-i\pi}y_2\right)^{\frac{1}{2}} \sum_{m,n\geq 0} B_{mn} y_1^m y_2^n \left(-\log(y_1) + \widetilde{\chi}_{mn}\right), \qquad (2.11)
\end{aligned}$$

where

$$\begin{aligned}
B_{mn} &= \frac{\Gamma(m+n+\frac{1}{2})^2}{\pi\Gamma(m+1)^2\Gamma(n+1)^2}\,, \\
\widetilde{\chi}_{mn} &= 2\Psi(m+1) - 2\Psi\left(m+n+\frac{1}{2}\right). \qquad (2.12)
\end{aligned}$$

It is clear that in order for $\widetilde{\Phi}_1$ and $\widetilde{\Phi}_2$ to be simultaneously real, we must have $|y_1| = 1$. Setting $y_2 = 0$, we find that the periods are simultaneously real. Thus, as expected from the dimension counting given above, we have found a one-dimensional real subspace where the gradings align.

The reader may be worried that $|y_1| = 1$ is right on the radius of convergence of the power series in $\widetilde{\Phi}_i$. One can show that the series converge for $y_1 \neq 1$. The divergence at $y_1 = 1$ is easy to understand: $y_1 = 1, y_2 = 0$ is on the discriminant locus of this model, which is given by

$$P(y_1, y_2) = 16y_1^2 - 8y_1y_2 + y_2^2 - 32y_1 - 8y_2 + 16\,. \qquad (2.13)$$

Unlike the case of the $\mathbb{P}^2$, here the grading align near the discriminant locus and not at the orbifold point. This is an important difference that is likely to persist for other del Pezzo surfaces. The example of $\mathbb{P}^1 \times \mathbb{P}^1$ is particularly tractable due to the symmetry between z_1 and z_2. The other toric del Pezzo surfaces can be treated in much the same fashion as above, but the computations are quite a bit more involved.

3. The category of B-branes on del Pezzo surfaces

3.1 Quivers and algebras

Before we can attack the problem of del Pezzo surfaces, we require some knowledge of the mathematics of quivers and tilting. See [4] and references therein for an account of the way quivers first appeared in the context at hand. We refer to [31] for a complete description of tilting. See also [21, 22] for accounts in the physics literature.

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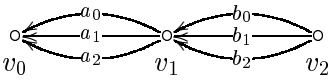
The first ingredient we require is the concept of the path algebra of a quiver. Let Q be a quiver with nodes v_i and arrows a_α. The path algebra A of Q is generated as follows. To each node v_i we associate an element e_i considered to be a path of length 0. The other generators consist of nonzero-length paths in the quiver. Clearly, each arrow a_α may be associated to a path. If the head of a_α is the same node as the tail of a_β, then we may produce a path $a_\beta a_\alpha$ consisting of a_α followed by a_β. Note that we compose paths right-to-left in our notation. This order is very important.

Multiplication in A is then defined as composition of paths in the obvious way. If the end of a path γ_1 is not the same node as the start of a path γ_2 then we define $\gamma_2\gamma_1 = 0$. Note that the zero-length paths e_i are idempotent: $e_i^2 = e_i$.

We can also impose *relations* on the quiver by asserting some relations the paths must obey. This amounts to setting A equal to some algebra generated by the paths divided by an ideal generated by the relations. For example, consider the following quiver



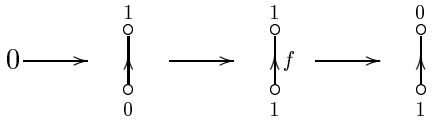
(3.1)

We could choose to (and will) impose the 3 relations $a_\alpha b_\beta = a_\beta b_\alpha$.

Let V be a given representation of the algebra A or, equivalently, a left A-module. Using the idempotent elements e_i, we form vector spaces $V_i = e_i V$. Let $N_i = \dim(V_i)$. The elements of A corresponding to arrows in Q then correspond to linear maps between the V_i's. That is, V can be associated to a set of integers n_i, one for each node, and a set of matrices, one for each arrow. Clearly these matrices will have to satisfy any relations that have been imposed on the quiver. This latter data is a "quiver representation." The inverse of this procedure is easily constructed (see [32], for example), showing that representations of A are equivalent to quiver representations of Q.

We may define a *morphism* between two representations W and V of A as a linear map $\phi : W \to V$ such that $a\phi(w) = \phi a(W)$ for any $w \in W$ and $a \in A$. Translating this into the language of quiver representations, this amounts to a set of linear maps $\phi_i : W_i \to V_i$ such that the $\phi_i' s$ commute with the maps within each quiver in the obvious way. Using the quiver representations as objects and the above morphisms, one defines the category of representations of Q (or equivalently left A-modules).

As a simple and useful example of a short exact sequence of quiver representations consider



(3.2)

The numbers in this diagram represent the dimensions n_i and f is multiplication by any complex number. The horizontal maps in this sequence between nontrivial vector spaces need not be zero. However, if f is nonzero, there are no nonzero morphisms going in the reverse directions to the ones shown.

There are two distinguished sets of useful quiver representation associated to a given quiver Q. In each case they are labeled by the nodes i. The first obvious set, L_i, corresponds to the one-dimensional representations given by $n_j = \delta_{ij}$. The second set, P_i, is defined

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by $P_i = Ae_i$. That is P_i is the subspace of A generated by all paths *starting* at node i. Multiplying on the left by elements of A makes P_i a left A-module and thus a representation.

Using $(N_0, N_1, \ldots)$ to denote the dimensions of representations, where $N_i = \dim(V_i)$ as above, it is easy to see in the example (3.1) that the dimensions are as follows

$$\begin{aligned}
\dim L_0 &= (1,0,0)\,, \\
\dim L_1 &= (0,1,0)\,, \\
\dim L_2 &= (0,0,1)\,, \\
\dim P_0 &= (1,0,0)\,, \\
\dim P_1 &= (3,1,0)\,, \\
\dim P_2 &= (6,3,1)\,.
\end{aligned} \tag{3.3}$$

One can show that the P_i are *projective* objects in the category of representations.

Note that if the quiver has any directed loops then some of the P_i's will be infinite-dimensional. This makes the analysis of such quivers considerably more difficult and most of the methods used in this paper will be useless. Luckily, by restricting our attention to del Pezzo surfaces we will effectively evade this case. From now on we will assume that there are *no* directed loops in the quiver Q.

Given that there are no directed loops in the quiver, we may assign an order to the labels of the nodes. We will assert that there is no path from node i to node j if $i < j$. This is consistent with our example (3.1).

It is not hard to show that the P_i's form a complete set of projective objects in the sense that any quiver representation has a *projective resolution* in terms of sums of P_i's. By using (3.3), one can see in our example that the following exact sequences form projective resolutions of the L_i's:

$$\begin{aligned}
0 \longrightarrow P_0 \longrightarrow L_0 \longrightarrow 0 \\
0 \longrightarrow P_0^{\oplus 3} \longrightarrow P_1 \longrightarrow L_1 \longrightarrow 0 \\
0 \longrightarrow P_0^{\oplus 3} \longrightarrow P_1^{\oplus 3} \longrightarrow P_2 \longrightarrow L_2 \longrightarrow 0
\end{aligned} \tag{3.4}$$

In general we will write the projective resolutions of the L_i representations as

$$\ldots \longrightarrow \bigoplus_k P_k^{\oplus r_{ik}} \longrightarrow \bigoplus_k P_k^{\oplus n_{ik}} \longrightarrow P_i \longrightarrow L_i \longrightarrow 0. \tag{3.5}$$

One can show that n_{ij} is equal to the number of arrows in the quiver from node i to node j and that r_{ij} represents the number of independent relations imposed on paths from i to j. Another basic fact about these representations which is easily proven is that

$$\mathrm{Hom}(P_i, L_j) = \delta_{ij}\mathbb{C}. \tag{3.6}$$

We would now like to compute some Ext groups which are central to our analysis. This is very easy in the current context. If an object A has a projective resolution

$$\cdots \longrightarrow \Pi_2 \longrightarrow \Pi_1 \longrightarrow \Pi_0 \longrightarrow A \longrightarrow 0, \tag{3.7}$$

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where the Π_i are projective objects (and thus direct sums of P_i's), then $\mathrm{Ext}^p(A,B)$ is given by the cohomology of the complex

$$0 \longrightarrow \mathrm{Hom}(\Pi_0, B) \longrightarrow \mathrm{Hom}(\Pi_1, B) \longrightarrow \mathrm{Hom}(\Pi_2, B) \longrightarrow \dots \tag{3.8}$$

in the pth position.

Thus we may use the resolutions (3.5) together with (3.6) to compute

$$\begin{aligned} \dim \mathrm{Ext}^1(L_i, L_j) &= n_{ij} \\ \dim \mathrm{Ext}^2(L_i, L_j) &= r_{ij} \,. \end{aligned} \tag{3.9}$$

One should therefore think of the arrows in a quiver as representing Ext^1's between the basic L_i representations and Ext^2's as arising because of relations in the quiver. Note that viewing the short exact sequence (3.2) provides another way of seeing that the arrows in a quiver correspond to Ext^1's.

The ordering we have chosen on the nodes implies that there can never be a non-zero map in the resolution $P_i \to P_j$ if $i > j$. This implies that $\mathrm{Ext}^p(L_i, L_j) = 0$ for any p if $i < j$.

We may now make contact with the quiver gauge theories of section 2. The arrows in the quiver gauge theory correspond to bifundamental chiral fields and are counted by Ext^1's between the D-branes in the derived category. *Therefore we associate the fractional D-branes at the nodes of quiver with the basic representations L_i.* The precise description for how composite D-branes decay into these fractional branes will be given in section 3.4.

3.2 Tilting

Given a quiver Q with path algebra A, let us denote the category of quiver representations (or left A-modules) by A–**mod**. We may now define the derived category[2] $\mathbf{D}(A\text{–}\mathbf{mod})$ to be the derived category obtained by passing to complexes of quiver representations. We refer the reader to [5] for what is intended to be a relatively gentle introduction to derived categories. Given another quiver with a path algebra B we would like to know when $\mathbf{D}(A\text{–}\mathbf{mod})$ is equivalent to $\mathbf{D}(B\text{–}\mathbf{mod})$.

If Z is a quiver representation, i.e., an object in A–**mod**, we will also use Z to denote an object in $\mathbf{D}(A\text{–}\mathbf{mod})$ consisting of a complex whose only nonzero entry is Z at position zero. With this notation in mind, consider a quiver Q with n nodes and define the object

$$T = P_0 \oplus P_1 \oplus \cdots \oplus P_{n-1} \,, \tag{3.10}$$

where the P_i's are the projective objects from section 3.1, and let

$$C = \mathrm{End}(T) \,, \tag{3.11}$$

i.e., C is the algebra of morphisms of T back to itself. Multiplication in this algebra is simply composition of morphisms. To each P_i we can clearly associate an idempotent

[2] All derived categories in this paper are bounded.

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element e_i in C corresponding to the projection of T onto P_i. Following carefully through the definition of morphisms above one can see that $\mathrm{Hom}(P_i, P_j)$ is given by the vector space of paths from j to i. This means that C is exactly the algebra one would associate to the quiver Q *if all the arrows were reversed*.

We may define the algebra $\mathrm{End}(T)^{\mathrm{op}}$ to be that given by $\mathrm{End}(T)$ except that the order of composition is reversed. This has the effect of reversing the direction of morphisms and thus we regain the original quiver. That is,

$$A \cong \mathrm{End}(T)^{\mathrm{op}} \,. \tag{3.12}$$

The idea of "tilting" is to replace T in (3.10) by a more general direct sum of objects satisfying particular conditions in order to get a new path algebra, distinct from A, but which is identical as far as derived categories are concerned.

To be precise, define a *complex* T to be an object in $\mathbf{D}(A\text{–}\mathbf{mod})$ such that

1. $\mathrm{Hom}(T, T[i]) = 0$ for $i \neq 0$.
2. The direct summands of T can be used to generate the whole of $\mathbf{D}(A\text{–}\mathbf{mod})$ by translations and mapping cones.

Here, as usual, we use the notation $T[i]$ to mean a left-shift of T by i places.[3] Mapping cones are the natural way of combining objects via a morphism in the derived category and correspond to combining D-branes via tachyon condensation. Again we refer to [5] for a review of this.

We then have following theorem due to Rickard [33] (following work by Happel [34])

Theorem 1. *The derived categories* $\mathbf{D}(A\text{–}\mathbf{mod})$ *and* $\mathbf{D}(B\text{–}\mathbf{mod})$ *are equivalent if and only if there exists a tilting complex* T *such that* $B = \mathrm{End}(T)^{op}$.

The tilting complex given by (3.10) clearly gives the equivalence of $\mathbf{D}(A\text{–}\mathbf{mod})$ to itself. It satisfies condition "1." as a tilting complex since $\mathrm{Ext}^p(P_i, P_j) = 0$ for $p > 0$ and condition "2." since all objects have a projective resolution.

3.3 Del Pezzo surfaces

Let S be a del Pezzo surface, i.e. a smooth surface whose anticanonical class intersects every algebraic curve in S a positive number of times, and let $i : S \to X$ be the embedding of this surface into a Calabi-Yau threefold X. Given an object in $\mathbf{D}(S)$, we may use the functor i_* to map this object into a D-brane in $\mathbf{D}(X)$. Physically this is the obvious statement that a B-brane in S may be viewed as a B-brane in X if S is embedded in X.

In order to describe B-branes wrapping on S, we need to describe the derived category $\mathbf{D}(S)$. Fortunately this is a well-known problem in algebraic geometry, and there are some very powerful tools established. In particular we may use the machinery of exceptional collections of sheaves (see [35] are references therein) based on Beilinson's [36] construction for $\mathbb{P}^n$.

[3] We will also, as usual, define $\mathrm{Ext}^p(A, B)$ to be $\mathrm{Hom}(A, B[p])$.

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Let $\{\mathscr{F}_0, \ldots, \mathscr{F}_{n-1}\}$ be an *exceptional collection* of sheaves on S. That is

$$\begin{aligned} \mathrm{Ext}^p_S(\mathscr{F}_i, \mathscr{F}_i) &= \begin{cases} \mathbb{C} & \text{if } p = 0, \\ 0 & \text{otherwise,} \end{cases} \\ \mathrm{Ext}^p_S(\mathscr{F}_i, \mathscr{F}_j) &= 0 \qquad \text{for any } p \text{ and } i > j. \end{aligned} \tag{3.13}$$

One can then prove [37] that, if $i < j$, then $\mathrm{Ext}^p(\mathscr{F}_i, \mathscr{F}_j)$ is nonzero for at most one value of p. A collection of sheaves is said to be *strongly* exceptional if $\mathrm{Ext}^p_S(\mathscr{F}_i, \mathscr{F}_j) = 0$ for $p \neq 0$. That is, only $\mathrm{Hom}_S(\mathscr{F}_i, \mathscr{F}_j)$ can be nonzero. An exceptional collection is said to be *complete* if it generates $\mathbf{D}(S)$. This latter condition is equivalent [37] to the number of elements in the exceptional collection being equal to the Euler characteristic of S.

If S is a del Pezzo surface dP_m with exceptional curves $C_1, C_2, \ldots C_m$, then a strongly exceptional collection is given by $\{\mathscr{O}, \mathscr{O}(C_1), \mathscr{O}(C_2), \ldots, \mathscr{O}(C_m), \mathscr{O}(H), \mathscr{O}(2H)\}$. Here H is a hyperplane $\mathbb{P}^1$ not intersecting any of the C_i's. Any exceptional collection may be obtained from this one by a sequence of mutations [37]. We will discuss mutations in section 4.2.

Assume $\{\mathscr{F}_0, \ldots, \mathscr{F}_{n-1}\}$ form a complete strongly exceptional set of sheaves on S and define

$$A = \mathrm{End}(\mathscr{F}_0 \oplus \mathscr{F}_1 \oplus \cdots \oplus \mathscr{F}_{n-1})^{\mathrm{op}}. \tag{3.14}$$

Bondal [38] then proved[4]

Theorem 2. *The derived category of coherent sheaves* $\mathbf{D}(S)$ *on* S *is equivalent to the derived category* $\mathbf{D}(A\text{–}\mathbf{mod})$.

Comparing this to (3.12) shows that the $\mathscr{F}_i$'s are playing the same rôle as the P_i's of section 3.2. In other words, the derived category of a del Pezzo surface S is equivalent to the derived category of representations of a quiver Q where the projective representations P_i correspond to a strongly exceptional set of sheaves.

This means that, given a strongly exceptional set of sheaves on S, we may construct the quiver immediately (and the set of relations) since we know that $\mathrm{Hom}_S(\mathscr{F}_i, \mathscr{F}_j)$ is precisely the space of paths from node j to node i. For example, consider the strongly exceptional collection $\{\mathscr{O}, \mathscr{O}(1), \mathscr{O}(2)\}$ on $S = \mathbb{P}^2$. Both $\mathrm{Hom}(\mathscr{O}, \mathscr{O}(1))$ and $\mathrm{Hom}(\mathscr{O}(1), \mathscr{O}(2))$ are given by $\mathbb{C}^3$ and correspond to multiplication by the homogeneous coordinates on $\mathbb{P}^2$. $\mathrm{Hom}(\mathscr{O}, \mathscr{O}(2)) \cong \mathbb{C}^6$ and is given by homogeneous quadratic function of the homogeneous coordinates. Any element of $\mathrm{Hom}(\mathscr{O}, \mathscr{O}(2))$ is given by an element of $\mathrm{Hom}(\mathscr{O}, \mathscr{O}(1))$ composed with an element of $\mathrm{Hom}(\mathscr{O}(1), \mathscr{O}(2))$, and thus no extra arrows are needed between node 0 and node 2. In addition, we have an obvious relation $x_i y_j = x_j y_i$ for the composition of such maps. Thus, after reversing the arrows in accord with the above description, we see that the quiver corresponding to $\mathbb{P}^2$ is given by our earlier example (3.1).

As another example, consider $S = \mathrm{dP}_1$ given by $\mathbb{P}^2$ with the single point $[z_0, z_1, z_2] = [0, 0, 1]$ blown up. Using the strongly exceptional collection $\{\mathscr{O}, \mathscr{O}(C_1), \mathscr{O}(H), \mathscr{O}(2H)\}$, the

[4] Bondal refers to *right* modules but these are turned into left modules by the "op" in (3.14).

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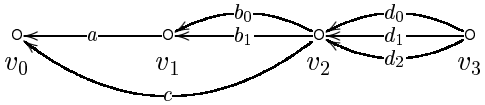
corresponding quiver is given by



(3.15)

subject to the relations $b_0d_1 - b_1d_0 = 0$, $ab_0d_2 - cd_0 = 0$, and $ab_1d_2 - cd_1 = 0$.

We emphasize that the sheaves in the exceptional collection are the projective objects P_i and not the fractional branes L_i. The relationship between these two sets of D-branes was given by projective resolutions (3.5). In references such as [11, 12] the exceptional sheaves themselves were taken to be the fractional branes. In [14] the sheaves corresponding to L_i were called a "dual collection" to the given collection P_i.

The physical problem we wish to analyze concerns D-branes on S embedded in X. Thus we need to apply the i_* map discussed at the start of this section. Objects in $\mathbf{D}(S)$ are mapped injectively to objects in $\mathbf{D}(X)$, however, there may be more morphisms between two objects in $\mathbf{D}(X)$ than there were in $\mathbf{D}(S)$. In other words, there are some open string states between two D-branes on S that live "outside" S in the threefold X.

Given two objects A and B in $\mathbf{D}(S)$, we may use a spectral sequence to compute the full spectrum of open strings $\mathrm{Ext}^m_X(i_*A, i_*B)$ as used in [39, 24], for example. Let N be the normal bundle to S in X. Then we have a spectral sequence with

$$E_2^{p,q} = \mathrm{Ext}^p_S(A, B \otimes \wedge^q N) \tag{3.16}$$

converging to $\mathrm{Ext}^{p+q}_X(i_*A, i_*B)$. In our case, because X is a Calabi-Yau manifold, N is equal to K_S, the canonical line bundle of S. Furthermore, Serre duality tells us that $\mathrm{Ext}^p_S(A, B \otimes K_S) = \mathrm{Ext}^{2-p}_S(B, A)$. The E_2 stage of our spectral sequence therefore looks like

$$\begin{array}{ccccccc}
\cdots & 0 & 0 & 0 & 0 & \cdots & \\
\cdots & \mathrm{Ext}^3_S(B,A) & \mathrm{Ext}^2_S(B,A) & \mathrm{Ext}^1_S(B,A) & \mathrm{Ext}^0_S(B,A) & \cdots & \\
\cdots & \mathrm{Ext}^{-1}_S(A,B) & \mathrm{Ext}^0_S(A,B) & \mathrm{Ext}^1_S(A,B) & \mathrm{Ext}^2_S(A,B) & \cdots &
\end{array} \tag{3.17}$$

(q vertical, p horizontal)

We have allowed for Ext^p's with $p < 0$ since we are working in the derived category.

Because we have no directed loops in the quiver associated to a del Pezzo surface, we may order the nodes and thus, as observed in section 3.3, either $\mathrm{Ext}^p(L_i, L_j)$ or $\mathrm{Ext}^p(L_j, L_i)$ must vanish for all p (except in the trivial case that $i = j$). Hence, only one row in (3.17) can contain nonzero entries, which immediately implies that there can be no d_2 or higher differentials in the spectral sequence. That is, the spectral sequence degenerates immediately to yield

$$\mathrm{Ext}^p_X(i_*L_i, i_*L_j) = \mathrm{Ext}^p_S(L_i, L_j) \oplus \mathrm{Ext}^{3-p}_S(L_j, L_i). \tag{3.18}$$

Actually there is a even stronger vanishing statement for the Ext's appearing in the spectral sequence. One can argue (see [13] for example) that the L_i's may be obtained from the given exceptional collection by a sequence of mutations. Corollary 2.11 of [37] states

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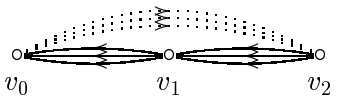
that $\mathrm{Ext}^p(L_i, L_j)$ will then be nonzero for at most one value of p. Thus, in the case that A and B are distinct fractional branes, at most one term in the diagram (3.17) is nonzero.

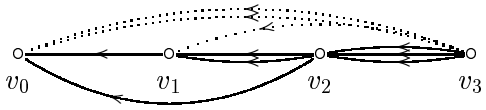
The gauge quiver for the del Pezzo has arrows corresponding to Ext^1's. Therefore, according to (3.18) we need to add arrows to the quiver corresponding to $\mathrm{Ext}^2_S(L_j, L_i)$ from node i to node j to account for the extra open strings induced by the embedding of S in X. As seen in section 3.1 these extra arrows are counted by the number of relations. Given an initial quiver Q, we will refer to the new quiver with the added arrows as the "completed quiver" and denote it $\bar{Q}$. The completed quiver is the gauge quiver.

The completed quiver for $\mathbb{P}^2$ therefore becomes



(3.19)

in agreement with the McKay quiver of $\mathbb{C}^3/\mathbb{Z}_3$, and the quiver for a dP_1 becomes



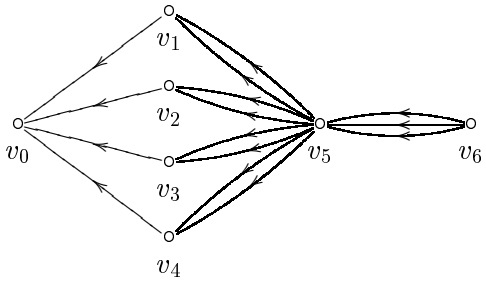
(3.20)

We will always use dotted arrows to represent the new arrows added in.

The completed quiver, once the extra arrows have been added in, contains oriented loops. This is always the case.[5] Because of this it is much harder to analyze the gauge quiver directly using techniques of quiver representations. The great thing about del Pezzo surfaces is that they allow us to consider a subcategory $\mathbf{D}(S)$ for which there are no oriented loops. Almost all of the time when we analyze quivers in this paper we will be treating the simpler loop-free quiver associated to $\mathbf{D}(S)$.

Equation (3.18) also shows us that $\mathrm{Ext}^3_S(B, A)$ will contribute to $\mathrm{Ext}^0_X(i_*A, i_*B)$. This is undesirable since Ext^0's correspond to tachyons by (2.2). In other words, if $\mathrm{Ext}^3(L_i, L_j)$ is nonzero for any pair of fractional branes then L_i and L_j will form a strongly bound state and completely rearrange the quiver in question. Thus, the prescription for determining the quiver from an exceptional collection breaks down.

There are cases where this problem occurs. For example, a strong complete exceptional collection on a dP_4 is given by $\{\mathscr{O}, \mathscr{O}(C_1), \mathscr{O}(C_2), \mathscr{O}(C_3), \mathscr{O}(C_4), \mathscr{O}(H), \mathscr{O}(2H)\}$. The quiver is then



(3.21)

[5] Eq. (3.18) implies that $\mathrm{Ext}^0_X(i_*L_j, i_*L_j) = \mathrm{Ext}^3_X(i_*L_j, i_*L_j) = \mathbb{C}$ for all j. The projective resolution of i_*L_j must therefore contain projective objects P_j in both the zero position and the third position ruling out any ordering of the nodes. Indeed, every node must be contained in at least one oriented loop.

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The projective resolution of L_6 is

$$0 \longrightarrow P_0 \longrightarrow P_1 \oplus P_2 \oplus P_3 \oplus P_4 \longrightarrow P_5^{\oplus 3} \longrightarrow P_6 \longrightarrow L_6 \longrightarrow 0, \tag{3.22}$$

and thus $\mathrm{Ext}^3(L_6, L_0) = \mathbb{C}$. The appearance of Ext^3's may also be viewed as "relations between the relations" in the quiver.

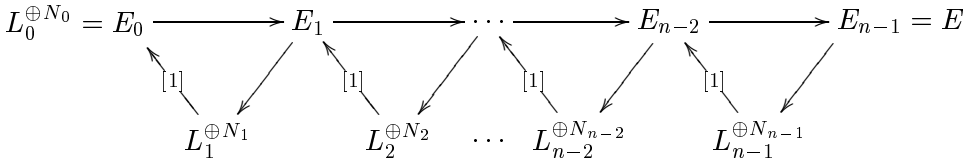
One way of evading these dangerous Ext^3's is to restrict attention to the "three block exceptional collections" of [40], as was done in [12]. This guarantees that no projective resolution of L_i need be more than 3 terms long, and thus the problematic Ext^3's are always zero. We should note, however, that the motivation for using three block exceptional collections given in [12], concerning canceling charges and anti-branes, does not apply when the derived category description is properly taken into account. There are also perfectly good non-three-block cases such as the dP_1 example we analyzed above.

In summary, given any strongly exceptional complete collection of sheaves on a del Pezzo surface we may construct the associated quiver gauge theory of fractional branes by the above methods if and only if $\mathrm{Ext}^p(L_i, L_j) = 0$ for all $p > 2$ and all pairs of fractional branes L_i and L_j.[6]

3.4 Decay into fractional branes

Suppose E is a quiver representation of dimension $(N_0, N_1, \ldots, N_{n-1})$. We may also view E as an object in the derived category $\mathbf{D}(S)$ as explained above.

Recall that, since there are no directed loops in the quiver associated to a del Pezzo surface, we may number the nodes such that there is no path from node i to node j if $i < j$. Once we number this way, we may construct the following set of distinguished triangles in $\mathbf{D}(S)$:



(3.23)

This chain also appeared in the work of Bridgeland [41] and was discussed in the context of D-brane decay in [5]. We now show that this chain of distinguished triangles exhibits how E decays into the collection of fractional branes $L_0^{\oplus N_0} \oplus L_1^{\oplus N_1} \oplus \cdots \oplus L_{n-1}^{\oplus N_{n-1}}$. The object E_k is a quiver representation with dimension $(N_0, N_1, \ldots, N_k, 0, 0, \ldots)$.

If, as we assume, the gradings of all the L_i are equal, then by the rules of Π-stability [26], $L_0^{\oplus N_0}$ and $L_1^{\oplus N_1}$ are marginally bound in the left-most triangle forming E_1 with the same grading ξ. This in turn implies that E_1 and $L_2^{\oplus N_2}$ are marginally bound forming E_2. We continue iteratively along the chain of triangles and see that E is a marginally bound state of the L_i's as desired.

This is the last ingredient we required to prove the following theorem:

[6] If, and only if, these conditions apply, then it follows that the intersection matrix inversion trick of dual exceptional collections as used in [13, 14] will also yield the gauge quiver.

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Theorem 3. *Suppose we are in a location in the $B + iJ$ moduli space corresponding to a collapsed del Pezzo surface S and where the gradings of the basis of fractional branes are aligned. Then the decay of composite D-branes is associated to a quiver gauge theory, where the quiver is given by the completion of a quiver associated to the path algebra* $\mathrm{End}(T)^{op}$ *and T is the sum of a strong complete exceptional collection of sheaves on S. The fractional branes are associated to the one-dimensional quiver representations L_i, where i labels the nodes in the quiver. The quiver gauge theory is tachyon-free if and only if* $\mathrm{Ext}^p(L_i, L_j) = 0$ *for all i and j and $p \geq 3$.*

The above discussion of decay has taken place in the context of $\mathbf{D}(S)$. What happens when we embed S into X? E is now associated to a quiver representation with extra arrows associated to Ext^2's. If this quiver representation associates nonzero matrices to any of these new arrows, then we will kill some of the morphisms used to construct the decay chain (3.23). If, on the other hand, we associate a zero matrix with each new arrow then nothing changes in our analysis above.

In other words, any quiver representation which associates nonzero matrices to the "new" arrows coming from Ext^2's, will not be marginally stable against a decay into the fractional branes. The interpretation is clear — turning on the matrices associated to the new arrows created by the embedding corresponds to moving the D-brane E away from S inside X. This was already understood in the case $S = \mathbb{P}^2$ [3].

The decay chain (3.23) allows one to compute the dimensions of a quiver representation of a 0-brane as follows. We may apply the functor $\mathrm{Hom}(P_k, -)$ to each of the distinguished triangles in the chain. From (3.6), starting from the left in the decay chain, this yields $\mathrm{Hom}(P_k, E_0) = 0 = \mathrm{Hom}(P_k, E_1) = \cdots = \mathrm{Hom}(P_k, E_{k-1}) = 0$. The next triangle gives $\mathrm{Hom}(P_k, E_k) = \mathbb{C}^{N_k}$. Continuing to the right finally yields

$$\mathrm{Hom}(P_k, E) = \mathbb{C}^{N_k} \,. \tag{3.24}$$

In our case, the projective objects P_k are given by exceptional sheaves $\mathscr{F}_k$. If E is a 0-brane then $E = \mathscr{O}_p$, the sky-scraper sheaf of a point. In this case $\mathrm{Hom}(P_k, E) = \mathrm{Hom}(\mathscr{F}_k, \mathscr{O}_p)$, where the latter is given by the rank of the sheaf $\mathscr{F}_k$. Thus we obtain

$$N_k = \mathrm{rank}(\mathscr{F}_k) \,. \tag{3.25}$$

In the examples above, the exceptional collections were comprised only of line bundles. The dimension of the quiver representation corresponding to a 0-brane is therefore $(1, 1, \ldots, 1)$ in each case.

Since $\mathscr{O}_p \otimes K_S = \mathscr{O}_p$, Serre duality tells us that $\mathrm{Ext}^n(A, E) \cong \mathrm{Ext}^{2-n}(E, A)$ for any A if E is the 0-brane. Defining, as usual

$$\chi_X(\mathscr{E}, \mathscr{F}) = \sum_i (-1) \dim \mathrm{Ext}^i_X(\mathscr{E}, \mathscr{F}) \,, \tag{3.26}$$

this implies, from (3.17), that $\chi_X(i_*A, i_*E) = 0$ for any object A in $\mathbf{D}(S)$. It follows [11, 12] that

$$\sum_i N_i(\bar{n}_{ik} - \bar{n}_{ki}) = 0 \,, \qquad \text{for all } k \,, \tag{3.27}$$

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where $\bar{n}_{ik}$ is the number of arrows from node i to node k in the *completed* gauge quiver (dotted arrows and all). This guarantees certain anomaly cancellations for the world-volume theory of the 0-brane [11].

In this paper we are principally concerned with marginal stability. However, one might also wish to know about the stability of E if we move away from our point of marginal stability. In this case, to leading oder, Π-stability becomes equivalent to the θ-stability of [42] as argued in [25]. Given the decay chain (3.23) one can argue [5] that the following may be deduced from the ordering of the gradings:

- $\xi(L_0) > \xi(L_1) > \cdots > \xi(L_{n-1})$ is a necessary and sufficient condition that E decays completely into $L_0 \oplus L_1 \oplus \cdots \oplus L_{n-1}$.
- If E is a single-term complex and the maps within the quiver are sufficiently generic (such as for $\mathcal{O}_x$), $\xi(L_0) < \xi(L_1) < \cdots < \xi(L_{n-1})$ is a sufficient (but not necessary) condition that E is stable.

4. Tilting duality

4.1 A class of tilts

In section 3.2 we considered the tilting procedure which allows one to demonstrate the equivalence of the derived category for different quivers. Actually we only considered the identity tilt of a quiver back to itself. We would now like to consider nontrivial tilts which, as observed in [21], correspond to Seiberg duality [20]. These tilts are similar to ones analyzed in [43].[7]

Define the following complex, denoted T_L, for a general quiver with no oriented cycles and n nodes. We choose a particular node k such that at least one arrow has its tail on this node, i.e., $n_{kj} \neq 0$ for some j.

$$T_L = \bigoplus_{j=0}^{n-1} P'_j, \tag{4.1}$$

where

$$\begin{aligned} P'_j &= P_j \quad \text{for } j \neq k, \\ P'_k &= \underline{\left(\textstyle\bigoplus_j P_j^{\oplus n_{kj}} \longrightarrow P_k \right)}, \end{aligned} \tag{4.2}$$

where the dotted underline means position zero in the complex. The morphism in the complex for P'_k is given by the natural composition of paths and, in particular, is nonzero.

T_L satisfies condition "2." for a tilting complex in section 3.2 since P_k can be obtained from P'_k from cone constructions to cancel out the added P_j's. Condition "1." is harder to prove. Let us introduce a little helpful notation. Let C be an object in the derived category given by a chain complex whose entries are C^i. Given two objects C and D in

[7] Except that [43] considers only nodes where the arrows are all incoming or all outgoing.

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the derived category we may produce $\hom(C,D)$ which is a complex of vector spaces with each entry given by

$$\hom^i(C,D) = \oplus_j \operatorname{Hom}(C^j, D^{j+i})\,, \tag{4.3}$$

and obvious differential maps in the complex. We refer to [44] for more details. The cohomology of this complex in the ith position is then $\operatorname{Ext}^i(C,D)$.

First note that, for $i \neq k$,

$$\begin{aligned}\hom(P'_i, P'_k) &= \big(\textstyle\bigoplus_j \operatorname{Hom}(P_i,P_j)^{\oplus n_{kj}} \longrightarrow \operatorname{Hom}(P_i,P_k)\big) \\ &= \big(\textstyle\bigoplus_j (\text{paths } j\to i)^{\#(\text{arrows } k\to j)} \longrightarrow (\text{paths } k\to i)\big)\,,\end{aligned} \tag{4.4}$$

where "(paths $j \to i$)" means the vector space generated by such paths. Since all paths from k to i must pass through a j in the sum, it is easy to see that the map in the latter complex is surjective. This implies that $\operatorname{Ext}^p(P'_i, P'_k) = 0$ for $p \neq 0$. Similarly

$$\begin{aligned}\hom(P'_k, P'_i) &= \big(\textstyle\bigoplus_j \operatorname{Hom}(P_j,P_i)^{\oplus n_{kj}} \xleftarrow{f} \operatorname{Hom}(P_k,P_i)\big) \\ &= \big(\textstyle\bigoplus_j (\text{paths } i\to j)^{\#(\text{arrows } k\to j)} \xleftarrow{f} (\text{paths } i\to k)\big)\end{aligned} \tag{4.5}$$

This map f (going from right to left) in the latter complex is given as follows. Given a path from i to k, compose it with each arrow from k to j to obtain a path from i to j. If this map were not injective it would imply that there is a nonzero linear combination of paths from i to k that, when composed with any nontrivial path starting at k, gives zero. Note that since we imposed $n_{kj} \neq 0$ for some j, there certainly are nontrivial paths starting at node k.

If the map f in (4.5) is injective we will call the object T_L "admissible" (following the language of [45]). In this case $\operatorname{Ext}^p(P'_k, P'_i) = 0$ for $p \neq 0$. It is easy to see in the case that all the objects are line bundles that T_L must be admissible. In the general higher rank case, this need no longer be true. The theory of mutations in section 4.2 can be used to prove that f must be either injective or surjective. These cases may be distinguished simply by computing the dimensions of the vector spaces involved. Thus, it is a simple matter to determine if a given T_L is admissible.

Finally, one can also show that $\hom(P'_k, P'_k) = \mathbb{C}$. If T_L is admissible, then combining the above results yields $\operatorname{Hom}(T_L, T_L[i]) = 0$ if $i \neq 0$, and thus T_L is a tilting complex.

The tilting recipe then tells us that we can construct an algebra $B = \operatorname{End}(T_L)^{\mathrm{op}}$. This is the path algebra of a new quiver Q' which has the same derived category as the original quiver. What is the new quiver? We need to compute $\operatorname{Ext}^p(L'_i, L'_j)$ where the L_i's are the new fractional branes for the quiver Q'.

In order to identify the L'_i's we require more details of how the tilting process yields an equivalence of derived categories. This is provided by a functor $\Psi_L : \mathbf{D}(A\text{–}\mathbf{mod}) \to \mathbf{D}(B\text{–}\mathbf{mod})$. If C is an object in $\mathbf{D}(A\text{–}\mathbf{mod})$, it can be shown that [31]

$$\Psi_L(C) = \mathbf{R}\operatorname{Hom}(T_L, C)\,. \tag{4.6}$$

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If E is an A-module, we can see that $\mathrm{Hom}_A(T_L, E)$ is a B-module as follows. Let h be an endomorphism of T_L and let f be a map from T_L to E. Then $h(f)$ is simply $f \circ h$.

Given (4.1), the new idempotent maps e'_i for the path algebra B of the quiver Q' are clearly projections of T_L to P'_i. Thus, if the dimension of $\Psi_L(E)$ is $(N'_0, N'_1, \ldots, N'_{n-1})$, then

$$\begin{aligned} N'_i &= \dim(e'_i \Psi_L(E)) \\ &= \dim \mathrm{Hom}(P'_i, E)\,. \end{aligned} \tag{4.7}$$

Here we have assumed that $\Psi_L(E)$ is a quiver representation rather than a complex of such representations, but the generalization to the derived category involves the usual manipulations.

Choose a node j of Q such that there are no arrows from k to j, i.e., $n_{kj} = 0$. From above it follows that $\Psi_L(L_j)$ has dimension given by $N'_i = \delta_{ij}$. In other words $\Psi_L(L_j) = L'_j$.

Next consider node k. The only nonzero $\hom(P'_i, L_k)$ is given by

$$\hom(P'_k, L_k) = \left(\mathbb{C} \longrightarrow \underset{\cdots}{0}\right). \tag{4.8}$$

This implies that $\Psi_L(L_k) = L'_k[1]$.

The remaining nodes in Q are nodes j such that $n_{kj} > 0$. In this case it is easy to check that the dimension of $\Psi_L(L_j)$ is the same as the dimension of $L'_j \oplus L_k'^{\oplus n_{kj}}$. This does not mean that $\Psi_L(L_j)$ really is isomorphic to $L'_j \oplus L_k'^{\oplus n_{kj}}$. Some of the matrices associated with any arrows between j and k in Q' may be nonzero for $\Psi_L(L_j)$. We will therefore use the more vague notation $L'_j \rightsquigarrow L_k'^{\oplus n_{kj}}$ for this object. In summary

$$\Psi_L(L_j) = \begin{cases} L'_k[1] & j = k \\ L'_j & j \neq k,\ n_{kj} = 0 \\ L'_j \rightsquigarrow L_k'^{\oplus n_{kj}} & j \neq k \end{cases} \tag{4.9}$$

Since the functor Ψ_L is an equivalence of categories, $\mathrm{Ext}^p(L_i, L_j) \cong \mathrm{Ext}^p(\Psi_L(L_i), \Psi_L(L_j))$. This, together with (4.9), is sufficient for us to compute the Ext groups between the L'_js and thus the new quiver Q'. In particular, if $n_{kj} = 0$ and $j \neq k$ then

$$\begin{aligned} \mathrm{Ext}^p(L'_j, L'_k) &= \mathrm{Ext}^{p-1}(L_j, L_k) \\ \mathrm{Ext}^p(L'_k, L'_j) &= \mathrm{Ext}^{p+1}(L_k, L_j) \end{aligned} \tag{4.10}$$

Next assume $n_{kj} > 0$ and hence $j < k$. Then $\mathrm{Ext}^p(L_j, L_k) = 0$ for any p and so $\mathrm{Ext}^{-1}(\Psi_L(L_j), \Psi_L(L_k)) = \mathrm{Hom}(\Psi_L(L_j), L_k) = 0$. Given that $\Psi_L(L_j)$ is $L'_j \rightsquigarrow L_k'^{\oplus n_{kj}}$, the only way that $\mathrm{Hom}(\Psi_L(L_j), L'_k)$ can be zero is if there is at least one arrow associated with a nonzero matrix from node j to k in Q'. There are no directed loops in Q' (since that would result in an infinite-dimensional B) and so there are no arrows from k to j in Q'. It follows that there is a short exact sequence

$$0 \longrightarrow L_k'^{\oplus n_{kj}} \longrightarrow \Psi_L(L_j) \longrightarrow L'_j \longrightarrow 0. \tag{4.11}$$

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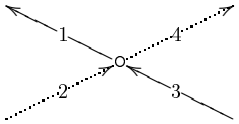
This gives a more precise description for $\Psi_L(L_j)$ than $L'_j \rightsquigarrow L'^{\oplus n_{kj}}_k$. Applying the functor $\hom(-, \Psi_L(L_k))$ to this sequence yields

$$\begin{aligned} \dim \mathrm{Ext}^1(L'_j, L'_k) &= n_{kj} \\ &= \dim \mathrm{Ext}^1(L_k, L_j)\,, \end{aligned} \tag{4.12}$$

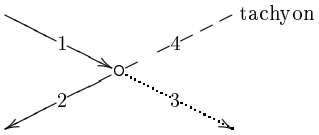
with all other Ext groups between L'_j and L'_k vanishing. This sequence also implies that the short exact sequence (4.11) is not split. This implies that the open string modes from L'_j to $L'^{\oplus n_{kj}}_k$ acquire nonzero values to form a "bound state" $\Psi_L(L_k)$.

We would now like to compare this tilting equivalence with Seiberg duality. We will consider the effect on the node k in the completed quivers $\bar{Q}$ and $\bar{Q}'$. Given our ordering, suppose we draw the nodes such that the labels increase from left to right. We then have the following types of arrows associated to node k:



(4.13)

Arrow 1 is associated to a $n_{kj} = 1$ and so (4.12) tells us that in $\bar{Q}'$ this will be reversed. Arrow 2 was associated to an $\mathrm{Ext}^2(L_k, L_j)$. According to (4.10), this will become $\mathrm{Ext}^1(L'_k, L'_j)$. That is, this arrow will flip and become solid. Arrow 3 will, by (4.10), become associated to an $\mathrm{Ext}^2(L'_j, L'_k)$ so will flip and become dotted. Finally arrow 4 was associated to $\mathrm{Ext}^2(L_j, L_k)$ and so becomes an $\mathrm{Ext}^3(L'_j, L'_k)$. As we saw in section 3.3, Ext^3's lead to tachyons and destroy our interpretation of the quiver. Thus the arrows in (4.13) for $\bar{Q}$, become, in $\bar{Q}'$:



(4.14)

So, if we impose the condition that there are no arrows of type 4, the tilting transformation takes a valid gauge quiver to another gauge quiver with no Ext^3's associated to node k and reverses all the arrows associated with node k. Furthermore, from (4.7), it is easy to compute

$$\begin{aligned} N'_j &= N_j, \quad j \neq k \\ N'_k &= \sum_i n_{ki} N_i - N_k\,. \end{aligned} \tag{4.15}$$

This transformation of the gauge groups together with the reversal of all arrows associated to node k is "Seiberg duality at node k." Even though insisting that there are no arrows of type 4 is sufficient to remove Ext^3's from node k in the new quiver, there is no guarantee that we don't induce new Ext^3's elsewhere in the quiver. This tends not to happen in simple examples but we will see an example of this occurrence in section 4.3.

Note that the ordering of the nodes in (4.14) is now broken, as arrow 1 points the wrong way. To regain the ordering, node k must be moved to the left of all nodes j for which $n_{kj} \neq 0$. This ties in with the language of mutations as we discuss in section 4.2.

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Imposing the condition that there are no arrows of type 4 implies that there must be arrows of type 1 from (3.27) and thus $n_{kj} \neq 0$ for some j. This latter condition, which was assumed at the start of this section, can therefore be subsumed by the condition that there are no arrows of type 4.

The arrows in the quiver away from node k will also be rearranged by this tilting transformation and computing the Ext^1's and Ext^2's for this transformation is straightforward using the above techniques. Note that we never need to compute the superpotential in order to perform the Seiberg duality.

Unlike what one might expect from Seiberg duality, the functor Ψ_L is not its own inverse. Indeed, if we were to apply Ψ_L again, then the arrows of type 3 would be tachyonic. Instead we define another functor Ψ_R given by the tilting complex

$$T_R = \bigoplus_{j=0}^{n-1} P_j'' \,, \tag{4.16}$$

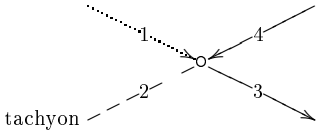
where

$$\begin{aligned} P_j'' &= P_j \qquad \text{for } j \neq k, \\ P_k'' &= \big(P_k \longrightarrow \textstyle\bigoplus_j P_j^{\oplus n_{jk}}\big). \end{aligned} \tag{4.17}$$

It follows that $\Psi_R(L_k) = L_k''[-1]$ and, if $j \neq k$, $\Psi_R(L_j)$ is determined by the short exact sequence

$$0 \longrightarrow L_j'' \longrightarrow \Psi_R(L_j) \longrightarrow L_k''^{\oplus n_{jk}} \longrightarrow 0. \tag{4.18}$$

Applying this transformation to the quiver piece (4.13) then yields



(4.19)

Thus the arrows are reversed if there are no arrows of type 2. We also have

$$\begin{aligned} N_j'' &= N_j \,, \qquad j \neq k \\ N_k'' &= \sum_i n_{ik} N_i - N_k \,. \end{aligned} \tag{4.20}$$

Note that, if the anomaly condition (3.27) is satisfied, and if there are no arrows of type 2 or 4, then $N_k'' = N_k'$, and Ψ_L and Ψ_R both equally generate Seiberg duality.

If we apply Ψ_L and then Ψ_R then

$$\begin{aligned} P_k'' &= \big(P_k' \longrightarrow \textstyle\bigoplus_j P_j'^{\oplus n'_{jk}}\big) \\ &= \mathrm{Cone}\big(P_k' \to \bigoplus_j P_j'^{\oplus n'_{jk}}\big) \\ &= \mathrm{Cone}\Big(\mathrm{Cone}\big(\bigoplus_j P_j^{\oplus n_{kj}} \to P_k\big)[-1] \to \bigoplus_j P_j^{\oplus n_{kj}}\Big) \\ &= P_k \,, \end{aligned} \tag{4.21}$$

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and so $\Psi_R \circ \Psi_L$ is the identity. Similarly $\Psi_L \circ \Psi_R$ is the identity too. Therefore, even though Seiberg duality is naïvely its own inverse, in the derived category picture it is given by two functors which are each other's inverse.

In summary, the effect of the tilting transformation Ψ_L for Seiberg duality is to replace L_k by $\Psi_L(L_k) = L'_k[1]$. Coarsely speaking, we replace L_k by its "anti-brane." The remaining L_j's are replaced by bound states $L'_j \leftrightsquigarrow L_k'^{\oplus n_{kj}}$ or, more precisely $\mathrm{Cone}(L'_j[-1] \to L_k'^{\oplus n_{kj}})$. This is reminiscent of descriptions for Seiberg duality elsewhere (see [46, 11] for example) but we believe that the derived category gives a much more precise picture as indicated in [21].

The tilting transformation Ψ_R also replaces L_k by an "anti-brane," but this time the shift is the "other way," as $\Psi_R(L_k) = L''_k[-1]$. The bound states also use the open strings in the opposite direction to Ψ_L. That is, L_j is replaced by $\mathrm{Cone}(L_k''^{\oplus n_{jk}}[-1] \to L''_j)$. By insisting on analyzing this problem using simplistic notions of anti-branes one would fail to distinguish between these two transformations.

4.2 Mutations

The functors Ψ_L and Ψ_R are equivalent to specific left and right *mutations* of the exceptional collection which makes contact with the work of [11, 9].

Suppose $\{\mathscr{F}_0, \ldots, \mathscr{F}_{n-1}\}$ corresponds to an exceptional collection of sheaves. In derived category language the left-mutation of $\mathscr{F}_k$ through $\mathscr{F}_{k-1}$ is defined by

$$\mathsf{L}_{\mathscr{F}_{k-1}}(\mathscr{F}_k) = \mathrm{Cone}\big(\hom(\mathscr{F}_{k-1}, \mathscr{F}_k) \otimes \mathscr{F}_{k-1} \to \mathscr{F}_k\big)[-1]\,. \tag{4.22}$$

Similarly, we define a right mutation:

$$\mathsf{R}_{\mathscr{F}_{k+1}}(\mathscr{F}_k) = \mathrm{Cone}\big(\mathscr{F}_k \to \hom(\mathscr{F}_k, \mathscr{F}_{k+1})^* \otimes \mathscr{F}_{k+1}\big)\,. \tag{4.23}$$

Now suppose $\mathscr{F}_0, \ldots, \mathscr{F}_{n-1}$ corresponds to a strongly exceptional collection of sheaves. Choose a number k and assume we have numbered the collection so that there is another number $l < k$ such that $\mathrm{Hom}(\mathscr{F}_j, \mathscr{F}_k)$ is nonzero if and only if $l \le j \le k$. Then the functor Ψ_L is equivalent to mutating $\mathscr{F}_k$ leftwards through members $k-1$ to l. Ψ_R is similarly constructed by right mutations.

In [37] it was shown that a left or right mutation in an exceptional collection of sheaves on a del Pezzo surface yields a complex with only one nonzero entry. Thus a mutation always yields an object of the form $\mathscr{E}[n]$ for some sheaf $\mathscr{E}$ and some integer n. The value of n is important (and distinguishes between "division," "extension," and "recoil" in mutation language [47]).

As a deceptively simple example consider $\{\mathscr{O}, \mathscr{O}(1), \mathscr{O}(2)\}$ as an exceptional collection on $\mathbb{P}^2$. We may reverse the order by mutating $\mathscr{O}(2)$ to the left twice and $\mathscr{O}(1)$ to the left once to obtain the exceptional collection $\{\mathscr{O}(-1), \Omega(1), \mathscr{O}\}$. In this case, no shifts are involved. Now consider $\{\mathscr{O}, \mathscr{O}(C_1), \mathscr{O}(H), \mathscr{O}(2H)\}$ as an exceptional collection on dP_1. Mutating the second entry to the left yields $\{\mathscr{O}_{C_1}(-1)[-1], \mathscr{O}, \mathscr{O}(H), \mathscr{O}(2H)\}$. The first entry is now shifted.

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It is interesting to note (as effectively observed in [45] and used in [13]) that the projective resolution of the L_k's can be phrased in terms of mutations. Up to shifts, one mutates L_k leftwards through all the members to the left. Carefully following through the derived category computation one finds that

$$L_k = \mathsf{L}_{P_0}\mathsf{L}_{P_1}\cdots\mathsf{L}_{P_{k-1}}P_k[k]\,. \tag{4.24}$$

It follows that all the fractional branes can be expressed as sheaves on the del Pezzo surface shifted by some integer. For example, the fractional branes on $\mathbb{P}^2$ are $\mathscr{O}$, $\Omega(1)[1]$ and $\mathscr{O}(-1)[2]$. Again we wish to emphasize the importance of taking the shifts into account. If one were merely to assert (as is commonly done) that the fractional branes were $\mathscr{O}$, anti-$\Omega(1)$, and $\mathscr{O}(-1)$, we know of no systematic computation that would correctly identify the spectrum of massless open strings between these D-branes. As explained in [5], the shifts are also important in seeing the quantum $\mathbb{Z}_3$ symmetry that arises in this particular example.

It is not true that a mutation necessarily yields a tilting and thus a Seiberg duality. The problem is that a strongly exceptional collection may be mutated into a non-strongly exceptional collection. This happened in section 4.1 if T_L or T_R was not admissible. A similar failure can happen if $\mathscr{F}_k$ is left-mutated through only some, but not all, of the $\mathscr{F}_j$'s with $\mathrm{Hom}(\mathscr{F}_j,\mathscr{F}_k)\neq 0$. For example, consider again the collection $\{\mathscr{O},\mathscr{O}(C),\mathscr{O}(H),\mathscr{O}(2H)\}$ on a dP_1. Mutating $\mathscr{O}(H)$ to the left of $\mathscr{O}(C)$ results in an exceptional collection which is not strong. We therefore do not have a quiver interpretation of this process. Mutating $\mathscr{O}(H)$ to the left of $\mathscr{O}(C)$ *and* $\mathscr{O}$ results in Seiberg duality on node v_2 in diagram (3.20).

Of course many mutations, even if they are a tilting equivalence, will produce Ext^3's and so will also not correspond to Seiberg duality. Given an exceptional collection, the sequence of mutations which reverses the order will typically produce Ext^3's. As we saw above, such a sequence of mutations converts from the P_i basis to the L_i basis. Therefore it is *not* generally true that these bases are Seiberg dual to each other.

The fact that an exceptional collection generates a "helix" [45] implies that a sequence of tilting transformations Ψ_L will result in the identity transform. Given an exceptional collection $\{\mathscr{F}_0,\ldots,\mathscr{F}_{n-1}\}$ on a del Pezzo surface we may mutate $\mathscr{F}_{n-1}$ leftwards through all $n-1$ elements on its left to produce a new element

$$\mathscr{F}_{-1} = \mathscr{F}_{n-1}\otimes K[3-n]\,, \tag{4.25}$$

where K is the canonical sheaf. Similarly we may now mutate $\mathscr{F}_{n-2}$ all the way to the left. Continuing this process until we have mutated $\mathscr{F}_0$ to the left, we end up with a new exceptional collection identical to the original one except that each element has been tensored with $K[3-n]$. This tensoring has no effect on the corresponding quiver and so we must return to the original quiver after all these transformations. There is, of course, a similar sequence of Ψ_R's that also produces the identity transform.

As an example, consider the quiver given by (3.20) for a dP_1. The sequence of tilting transforms given by Ψ_L associated to the above helix is shown in figure 1. The node (counting from left to right starting at 0) to which Seiberg duality is applied is shown over

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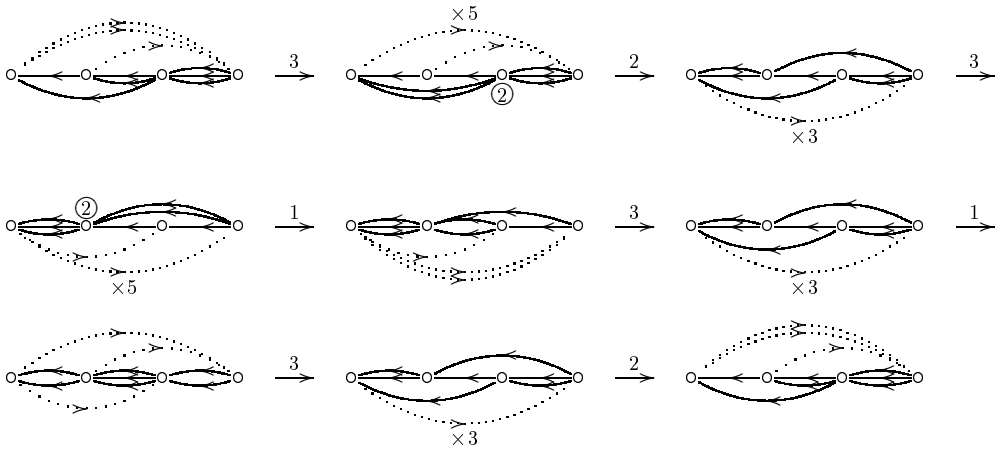
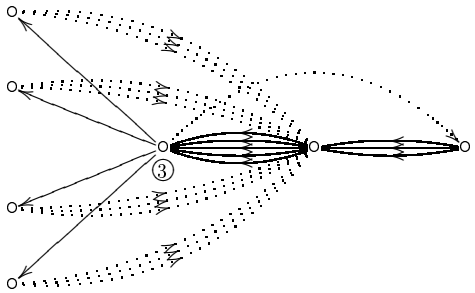


Figure 1: A sequence of Ψ_L transforms equivalent to the identity for a dP_1.

the arrows between the diagrams. A number in a circle gives that value of N_i (i.e., the rank of the gauge group) for that node (the default being 1). After applying the tilting transformation we have reordered the nodes to get back to an ordered graph. This is exactly the reordering process familiar in mutations. All our diagrams will use this reordering from now on. Note that in figure 1 we have certain duplications appearing in the sequence, but this is not a general feature in more complicated examples.

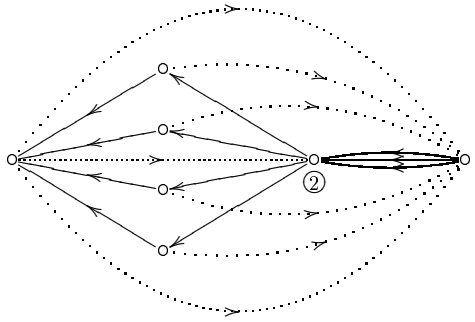
4.3 Removing Ext^3's

As well as producing a Seiberg duality, we may also use the tilting of section 4.1 to remove unwanted Ext^3's from quivers. For example, in section 3.3 we considered the quiver (3.21) associated to a dP_4 for which $\mathrm{Ext}^3(L_6, L_0)$ is nonzero. By applying Ψ_R to node 0 we convert this to an Ext^2 yielding the quiver



(4.26)

Alternatively we could apply Ψ_L to node 6 to produce another valid quiver:



(4.27)

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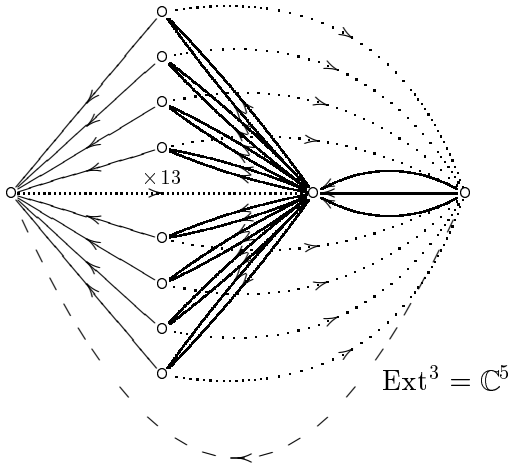
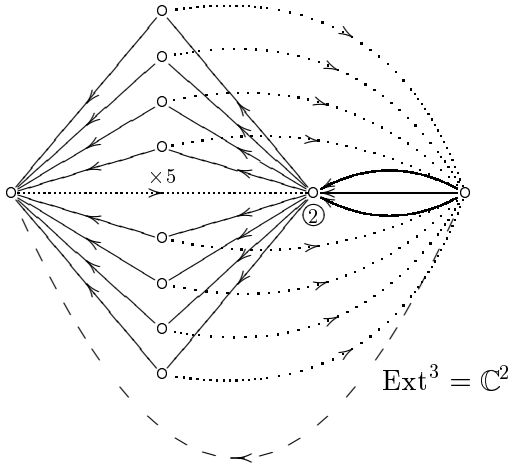
The nonzero Ext^3 in the original quiver induced a tachyon, and so one would expect the field theory to seek out a true vacuum by giving an expectation value to this mode. The two quivers above are not expected to be the new vacuum. This is because, as explained in section 4.1, the tiltings are obtained by giving expectation values to other arrows in the quiver (corresponding to massless modes for our chosen point in moduli space). It would be interesting to determine the quiver that results from the tachyon field acquiring a vev.

It is possible that applying Ψ_L or Ψ_R can eliminate unwanted Ext^3's from node k, but this procedure can induce new Ext^3's elsewhere. As an example, consider the exceptional collection $\{\mathscr{O}, \mathscr{O}(C_1), \ldots, \mathscr{O}(C_8), \mathscr{O}(H), \mathscr{O}(2H)\}$ on a dP_8. This has a quiver



(4.28)

Applying Ψ_L to the right-most node we may remove the problematic Ext^3. The resulting quiver is now



(4.29)

Thus, we gained some new Ext^3's resulting in another invalid quiver.

4.4 Some conjectures

Having explored a large number of tilting transformations we would like to make the following conjectures which seem to always hold. All of these conjectures are essentially statements about linear algebra so there may well be a simple proof that we have missed.

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Let us say that T_L is a "valid" tilting complex for a given node if there are no arrows of type 4, and T_R is valid if there are no arrows of type 2.

Conjecture 1.

1. *All valid T's are admissible (in the sense of section 4.1).*
2. *If T_L and T_R are both valid for a given node then they yield the same completed quiver (making no distinction between* Ext^1*'s and* Ext^2*'s).*
3. *For any node, either T_L or T_R is valid (i.e., we never have arrows of type 2 and 4 on a particular node).*
4. *A valid T applied to a quiver free of* Ext^3*'s will not induce any new* Ext^3*'s.*

Assuming these conjectures are correct, Seiberg duality is straight-forward for quivers. We are free to apply duality to any node and we obtain a unique valid result.

5. Conclusions

We have shown in detail that a 0-brane on a collapsed del Pezzo surface decays marginally into a set of fractional branes in a way described by a quiver. The B-type D-branes on this del Pezzo surface are then described by the derived category of representations of this quiver. This allowed us to describe a very large class of Seiberg dualities using the language of tilting equivalences.

It is very important to distinguish between Ext^1's and Ext^3's between D-branes. The former give massless bifundamental chiral fields, while the latter give tachyons. Any analysis which only uses intersection pairings at the K-theory level (such as $\chi(A, B)$ or, in the mirror, intersection numbers of 3-cycles) will miss this distinction. Fortunately the derived category provides a rigorous and complete framework for understanding this problem.

The general program of trying to use tilting equivalences to analyze Seiberg duality is frustrated by the appearance of infinite-dimensional representations caused by oriented loops in the quiver. In the case of del Pezzo surfaces this problem is avoided and the tilting transformations become equivalent to mutations of exceptional collections of sheaves. It may well be possible to deal directly with the quivers containing loops, as was done in [22]. It may also provide some insight into the conjectures of section 4.4.

The obvious extension of our work would be to consider *generalized* del Pezzo surfaces and to consider more than one such surface collapsing to the same point. This would provide a general analysis of all singularities that can occur at a point which can be blown-up by an exceptional divisor. This would, of course, include all orbifolds but would provide a much larger class. It would be interesting to see if representations of quiver path algebras would continue to play a useful rôle in this wider problem. Some results on weighted projective space along this line are discussed in [48, 49].

We have had very little to say about the superpotential in this paper. Indeed, it is quite interesting to note that the tilting transformations allow one to compute Seiberg

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dualities without knowing the superpotential. Having said that, the derived category of quiver representations does provide all information about the superpotential in the form of A_∞ algebras along the lines of [50]. Presumably this ties in with the work already done on superpotentials in this subject [11, 12]. The arena of del Pezzo surfaces should provide many nice examples for studying the A_∞ structure.

Acknowledgments

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